

STABILITY OF KÄHLER-RICCI FLOW IN THE SPACE OF KÄHLER METRICS

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ABSTRACT. In this paper, we prove that on a Fano manifold M which admits a Kähler-Ricci soliton (ω, X) , if the initial Kähler metric ω_{φ_0} is close to ω in some weak sense, then the weak Kähler-Ricci flow exists globally and converges in Cheeger-Gromov sense. Moreover, if φ_0 is also K_X -invariant, then the weak modified Kähler-Ricci flow converges exponentially to a unique Kähler-Ricci soliton nearby. Especially, if the Futaki invariant vanishes, we may delete the K_X -invariant assumption. The methods based on the metric geometry of the space of the Kähler metrics are potentially applicable to other stability problem of geometric flow near a critical metric.

1. INTRODUCTION

Ricci flow, introduced by Hamilton [23], plays an important role in understanding the geometric and topological structure of the manifolds which it lives on. We call the Ricci flow a Kähler-Ricci flow, if the underlying manifold is a Kähler manifold. Furthermore, the normalized Kähler-Ricci flow is given by

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} \omega &= -Ric + \lambda \omega, \\ \omega(0) &= \omega_{\varphi_0} \end{cases}$$

in which $\omega(0)$ stays in the canonical class $2\pi C_1(M)$ and λ is the sign of the first Chern class. Cao [7] first showed that Kähler-Ricci flow (1.1) has long time existence and converges to a Kähler-Einstein metric when the first Chern class is negative or zero. Now we restrict ourselves in the situation that the first Chern class is positive. Since the Kähler-Ricci flow preserves the Kähler class, we rewrite the Kähler-Ricci flow in the potential level as

$$(1.2) \quad \begin{cases} \frac{\partial \varphi}{\partial t} &= \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi - h_{\omega} + a(t), \\ \varphi(0) &= \varphi_0 \end{cases}$$

where $a(t)$ is a constant depending on t and h_{ω} is the Ricci potential of the reference metric ω defined by

$$(1.3) \quad \sqrt{-1} \partial \bar{\partial} h_{\omega} = Ric(\omega) - \omega \text{ and } \int_M e^{h_{\omega}} \omega^n = Vol(M).$$

In Perelman [30], he obtained an estimate of the Kähler-Ricci flow and proved that the Kähler-Ricci flow converges to a Kähler-Einstein metric in the sense of Cheeger-Gromov when one exists for any initial Kähler metric. Later on, Sesum-Tian [34] gave the detailed proof. Furthermore, Tian-Zhu [40] extended it to the case of Kähler-Ricci soliton for a K_X -invariant initial metric. In which, a Kähler-Ricci

soliton is a Kähler metric such that if there is a holomorphic vector field X such that

$$(1.4) \quad L_X \omega = Ric - \omega.$$

Since the right side of the equation (1.4) is real-valued, we obtain $L_{\Im X} \omega = 0$ and $\Im X$, the imaginary part of X , generates a one-parameter isometry group K_X .

In order to study the asymptotic behavior of the Kähler-Ricci flow, we consider the stability problem. I.e. on a Kähler manifold M admits a Kähler-Ricci soliton (ω, X) , in what kind of neighborhood of ω , the Kähler-Ricci flow with its initial datum stays, converges in some sense (maybe exponentially) to a Kähler-Ricci soliton.

This stability problem of the Kähler-Ricci flow has been initiated and studied by many people, for complete references we refer to Chen-Li [12]. In Chen-Li [12] and Tian-Zhu [41], they consider perturbing both the initial metric and the complex structure near a Kähler-Einstein metric.

In this paper, we focus on perturbing the initial metric near the Kähler-Ricci soliton without changing the complex structure. Firstly, we give a direct proof of the long time existence and convergence in Cheeger-Gromov sense within the frame of Donaldson's programme [18]. The proof of which is based on the geometry of the space of Kähler metrics. Next, we derive the exponential convergence and the uniqueness of the limit via calculating the energy function. Set $\mathcal{N}(\epsilon_0; B, p)$ be a small neighborhood of the zero function depends on ϵ_0 , B and p which will be specified in Section 7. The main results of this paper are given as follows:

Theorem 1.1. *On a manifold admits a Kähler-Ricci soliton (ω, X) , there exists a positive constant ϵ_0 , if the initial potential φ_0 stays in $\mathcal{N}(\epsilon_0; B, p)$, then the weak Kähler-Ricci flow exists globally and converges in Cheeger-Gromov sense. Moreover, if φ_0 is K_X -invariant, the weak modified Kähler-Ricci flow converges exponentially to a unique Kähler-Ricci soliton nearby.*

When the Futaki invariant vanishes, it is obvious that the holomorphic vector fields $X = 0$ and the Kähler-Ricci soliton is a Kähler-Einstein metric, then we have

Theorem 1.2. *On a Kähler-Einstein manifold, there exists a positive constant ϵ_0 , if the initial potential φ_0 stays in $\mathcal{N}(\epsilon_0; B, p)$, then the weak Kähler-Ricci flow exists globally and converges exponentially to a unique Kähler-Einstein metric nearby.*

Simon [35] studied the asymptotic behavior of the gradient flow of the variation problem by so called the Łojasiewicz-Simon inequality which compares the distance to the critical set with the norm of the gradient of the functional in the L^2 space under the condition that the functional should be analytic. The underlying idea is to reduce the infinite-dimensional problem to a finite-dimensional problem. Perelman [31] introduced a new functional called μ functional and pointed out that the Ricci flow is the gradient flow of the μ functional up to a diffeomorphism.

However, in this paper, we do not apply the Łojasiewicz-Simon inequality to the μ functional directly. In fact, we provide a new approach to study the asymptotic behavior of the flow which is even only a pseudo-gradient flow of some functional, since in Kähler setting, geometry gives us more information. To be precise, the critical set in the space of Kähler metrics is a finite dimension Riemannian symmetric space we will explained later.

Since the Kähler-Ricci flow is the pseudo-gradient flow of the K -energy, in order to make the mechanism of our proof more clear, we firstly prove Theorem 1.2 under the assumption that the $C^{2,\alpha}$ norm of φ_0 is small. Furthermore, we generalize our approach to the case of Kähler-Ricci soliton, Theorem 1.1.

We sketch our proof of Theorem 1.1 and Theorem 1.2 as follows. We first prove the Kähler-Ricci flow (1.2) after pulling back by the corresponding holomorphic transformations will always stay in a small neighborhood near the background Kähler-Einstein metric. When M has no nontrivial holomorphic vector fields, it is not necessary to find the transformations and in Section 3 the proof of which is given. However, in general, when M admits nontrivial holomorphic vector fields, in Section 4.1 we develop a new method to pick up the appropriate transformations following the trace of the Kähler-Ricci flow in the space of normalized Kähler potential \mathcal{H}_0 (c.f. (2.2)). It has been shown by Mabuchi [28], Donaldson [17] and Semmes [33] independently that \mathcal{H}_0 is a infinite dimensional symmetry space of negative curvature. Later, Chen [10] proved \mathcal{H}_0 is also a metric space. Since the space of potentials of Kähler-Einstein metrics, \mathcal{E}_0 , is a totally geodesic submanifold in \mathcal{H}_0 , the projection ρ minimizing the distance function from the Kähler-Ricci flow to \mathcal{E}_0 is uniquely determined. The Bando-Mabuchi's uniqueness theorem of Kähler-Einstein metric [3] implies ω_ρ is different from the reference Kähler-Einstein metric by a holomorphic transformation. The projection Kähler-Einstein metric is exactly the new reference metric we acquired.

Another way to derive a holomorphic transformation (in the Appendix) of $\varphi \in \mathcal{H}_0$ is to minimize the $I - J$ functional in \mathcal{E}_0 , which has been introduced by Bando-Mabuchi [3] to prove the uniqueness of the Kähler-Einstein metric. However, their method can not be applied in our case directly, since in general the hessian of $I - J$ functional is not strictly positive, i.e. the minimizer is not unique. Nevertheless, as we observed when the $C^{2,\alpha}$ norm of φ is small, the hessian of $I - J$ functional is indeed strictly positive. Therefore, the holomorphic transformation is uniquely determined.

Next, in Section 5, we derive the exponential convergence of the Kähler-Ricci flow by computing the energy functions and using the Futaki invariant. The key idea is since the geometric quantities such as the Sobolev constant and the Poincare constant are invariant under the holomorphic transformation, the De Giorgi-Nash-Moser iteration can be applied to control $\frac{\partial \varphi}{\partial t}$.

Then in Section 6, we prove a stability theorem of Kähler-Ricci flow near a Kähler-Ricci soliton (ω, X) similarly to the case of Kähler-Einstein metric. We first prove the Kähler-Ricci flow (1.2) modulo automorphisms will always stay in a small neighborhood near the background Kähler-Ricci soliton for arbitrary initial Kähler potential with small $C^{2,\alpha}$ norm. The key idea is to use Perelman's μ functional [31] instead of the K -energy, since the hessian of the μ functional is nonnegative at a Kähler-Ricci soliton within the canonical class [41]. Furthermore, we reparametrize the Kähler-Ricci flow (1.1) by the automorphisms $\varsigma(t)$ generated by the real part $\Re X$ of X such that

$$(1.5) \quad \begin{cases} \frac{\partial}{\partial t} \omega_\phi &= -Ric(\omega_\phi) + \omega_\phi + L_{\Re X} \omega_\phi, \\ \omega_{\phi(0)} &= \omega_{\varphi_0}. \end{cases}$$

It is obvious that the Kähler-Ricci soliton is the stationary solution of the modified Kähler-Ricci flow (1.5). Since the Kähler-Ricci soliton (ω, X) is K_X -invariant

and the Kähler-Ricci flow is also invariant under the holomorphic diffeomorphism, without loss of generality, we assume the initial datum is K_X -invariant. Then we generalize the exponential convergence of the Kähler-Ricci flow derived in Section 5 to the modified Kähler-Ricci flow (1.5).

Finally, in Section 7, at a fixed time, we show that the $C^{2,\alpha}$ norm of the potential is small when the initial value is small under certain weak condition. The main idea is to use the estimate introduced in [16].

As a corollary of Theorem 1.1, we deduce that the limit metric of the Kähler-Ricci flow is unique. Set $\{\varphi(t_i)\}$ be a sequence of the solution of the Kähler-Ricci flow which converges to a Kähler-Einstein metric or Kähler-Ricci soliton g_∞ , if there exists, then there exists some $\varphi \in \{\varphi(t_i)\}$ such to the stability-condition given in Theorem 1.1. According to the stability Theorem 1.1, the Kähler-Ricci flow with initial-value φ converges exponentially to a Kähler-Einstein metric g_∞^1 (or Kähler-Ricci soliton respectively). Furthermore, since we assume that $\{\varphi(t_i)\} \rightarrow g_\infty$, so g_∞^1 must coincide with g_∞ .

We emphasize that our approach using to prove Theorem 1.1 is also applicable to the case for the general pseudo-gradient flow. I.e. neither the condition “flow is a gradient flow of some functional”, the Perelman’s deep estimate [30], nor a prior long time existence of the flow is required. It is possible that our method can be utilized to solve similar problem of other geometric flow problems. For instance, to prove the stability theorem of the pseudo-Calabi flow near a constant scalar curvature Kähler (cscK) metric in [11] and of the Calabi flow near a extremal metric in [24].

The paper is organized as follows: In section 2 we review the known results of the space of Kähler metrics and the well-posedness of the pseudo-Calabi flow (c.f. (2.11)) we obtain in [11]. In Section 3, Section 4 and Section 5 we first prove theorem Theorem 1.2 under the assumption that the $C^{2,\alpha}$ norm of the initial Kähler potential is small. Then we prove Theorem 1.1 under the same assumption in Section 6. Finally, in Section 7 we explain how to weaken the initial condition to which stated in both Theorem 1.2 and Theorem 1.1. In the Section 8, we explain another method to choose the holomorphic transformation.

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2. NOTATIONS AND BASIC RESULTS

Let M be a compact Kähler manifold of complex dimension n with positive first Chern class $C_1(M)$ and ω be a Kähler form which represents the canonical class $2\pi C_1(M)$. In a local holomorphic coordinate z_1, z_2, \dots, z_n , ω is expressed by

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

The corresponding Riemannian metric is given by

$$g = \sum_{i=1}^n g_{i\bar{j}} dz^i \otimes dz^{\bar{j}}.$$

For a Kähler metric ω , the volume form is

$$dV = \omega^n = (\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

The Ricci form taking the form

$$Ric = \sqrt{-1} \sum_{i=1}^n R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = -\sqrt{-1} \partial \bar{\partial} \log \det \omega^n$$

is a closed real $(1, 1)$ -form and stays in $2\pi C_1(M)$. According to which, we obtain the scalar curvature satisfies

$$S\omega^n = n Ric \wedge \omega^{n-1}.$$

Furthermore, a direct calculation gives the average of the scalar curvature

$$\underline{S} = \frac{1}{V} \int_M S dV = \frac{n}{V} \int_M Ric \wedge \omega^{n-1} = n.$$

Let \mathcal{K} be the set of all Kähler forms on M representing $2\pi C_1(M)$ and \mathcal{E} be the set of all Kähler-Einstein metrics in \mathcal{K} . According to $\partial\bar{\partial}$ lemma, for any Kähler metric ω' in \mathcal{K} there exists a smooth real-valued function φ such that $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. Then the space of Kähler potentials of \mathcal{K} is given by

$$\mathcal{H} = \{\varphi \in C^\infty(M, \mathbb{R}) | \omega + \sqrt{-1}\partial\bar{\partial}\varphi \in \mathcal{K}\}.$$

Apparently, we have a isomorphism $T\mathcal{H} \cong \mathcal{H} \times C^\infty(M, \mathbb{R})$. Mabuchi [28], Donaldson [17] and Semmes [33] independently defined a Riemannian metric on \mathcal{H} by

$$\int_M f_1 f_2 \omega_\varphi^n$$

for any $f_1, f_2 \in T_\varphi \mathcal{H}$. For any path $\varphi(t) (0 \leq t \leq 1)$ in \mathcal{H} , the length is given by

$$(2.1) \quad L(\varphi(t)) = \int_0^1 \sqrt{\int_M \varphi'(t)^2 \omega_{\varphi(t)}^n} dt$$

and the geodesic equation is

$$\varphi''(t) - \frac{1}{2} |\nabla_t \varphi'(t)|_{\varphi(t)}^2 = 0$$

in which we use $'$ to denote the differentiation in t and ∇_t to denote the covariant derivative for the metric $g_{\varphi(t)}$. The geodesic equation enables us to define the connection on the tangent bundle. For any tangent vector field $\psi(t)$ along the path $\varphi(t)$, the covariant derivative along $\varphi(t)$ is defined by

$$D_t \psi = \frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla_t \psi, \nabla_t \varphi')_{g_\varphi}.$$

Then the connection at φ is given by

$$\Gamma(\psi_1, \psi_2) = -\frac{1}{2} (\nabla \psi_1, \nabla \psi_2)_{g_\varphi}$$

for any ψ_1 and ψ_2 in $T_\varphi \mathcal{H}$. Moreover, Γ is torsion-free and metric-compatible. The following theorem is proved in [28], [17] and [33].

Theorem 2.1. (Mabuchi [28], Donaldson [17], Semmes [33]) *The Riemannian manifold \mathcal{H} is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\varphi \in \mathcal{H}$ the curvature is given by*

$$R_\varphi(\delta_1\varphi, \delta_2\varphi)\delta_3\varphi = -\frac{1}{4}\{\{\delta_1\varphi, \delta_2\varphi\}_\varphi, \delta_3\varphi\}_\varphi,$$

where $\{\cdot, \cdot\}_\varphi$ is the Poisson bracket on $C^\infty(M)$ of the symplectic form ω_φ .

Chen established the following theorem in [10].

Theorem 2.2. (Chen [10]) *The following is true:*

- (i) \mathcal{H} is convex by $C^{1,1}$ geodesics.
- (ii) \mathcal{H} is a metric space.

Later, Calabi and Chen proved \mathcal{H} is negatively curved in the sense of Alexanderof in [6]. We denote the space of normalized Kähler potentials by

$$(2.2) \quad \mathcal{H}_0 = \{\varphi \in C^\infty(M, \mathbf{R}) \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ and } I(\varphi) = 0\},$$

where

$$I(\varphi) = \frac{1}{V} \sum_{p=0}^n \frac{1}{(p+1)!(n-p)!} \int_M \varphi \omega^{n-p} \wedge (\partial\bar{\partial}\varphi)^p.$$

In fact, \mathcal{H} can be naturally split as

$$\mathcal{H} = \mathcal{H}_0 \times \mathbf{R}.$$

It leads to the decomposition of the tangent space

$$T_\varphi = \{f \mid \int_M f \omega_\varphi^n = 0\} \oplus \mathbf{R}.$$

On a Kähler-Einstein manifold (M, ω) , choose ω be the reference metric. It is clear that $h_\omega = 0$ by the definition (1.3). Substituting this into the potential equation of Kähler-Ricci flow (1.2), we obtain that

$$(2.3) \quad \begin{cases} \frac{\partial \varphi}{\partial t} &= \log \frac{\omega_\varphi^n}{\omega^n} + \varphi + a(t) \\ \varphi(0) &= \varphi_0. \end{cases}$$

Furthermore, we choose appropriate normalization constant

$$(2.4) \quad a(t) = -\frac{1}{V} \int_M (\log \frac{\omega_\varphi^n}{\omega^n} + \varphi) \omega_\varphi^n,$$

then one obviously sees that

$$(2.5) \quad \partial_t I(\varphi) = \frac{1}{V} \int_M \partial_t \varphi \omega_\varphi^n = 0.$$

We first assume $\varphi_0 \in \mathcal{H}_0$ such that $I(\varphi_0) = 0$, the general case will be treated in Section 7. Then (2.5) implies $I(\varphi) = 0$ which assures the solution φ of (2.3) always stays in \mathcal{H}_0 .

For any $\varphi \in \mathcal{H}$, Mabuchi [27] defined the K -energy of (M, ω) as follows

$$(2.6) \quad \nu(\omega, \omega_\varphi) = -\frac{1}{V} \int_0^1 \int_M \dot{\varphi}(\tau) (S_{\varphi(\tau)} - \underline{S}) \omega_{\varphi(\tau)}^n d\tau$$

where $\varphi(\tau)$ is an arbitrary piecewise smooth path from 0 to φ . Later on, the explicit expression of the K -energy is given in Chen [9] and Tian [37] as

$$(2.7) \quad \begin{aligned} \nu_\omega(\varphi) &= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \frac{Sn!}{V} I(\varphi) \\ &\quad - \frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_M \varphi Ric \wedge \omega^{n-1-i} \wedge (\partial\bar{\partial}\varphi)^i. \end{aligned}$$

We will in later section simply denote $\nu(\varphi)$ instead of $\nu_\omega(\varphi)$. The second variation of the K -energy is given in Mabuchi [28].

Theorem 2.3. (Mabuchi [28]) *If ω is a critical point of $\nu(\varphi)$, then the inequality*

$$(2.8) \quad \frac{d^2}{dt^2} \nu(\theta_t)|_{t=0} \geq 0$$

holds for every smooth path $\{\theta_t | -\epsilon \leq t \leq \epsilon\}$ in \mathcal{K} such $\theta_0 = \omega$.

Let $Aut(M)$ be the group of holomorphic automorphisms of M and $Aut_0(M)$ be its identity component. Bando-Mabuchi [3] and Bando [2] further showed that

Theorem 2.4. (Bando-Mabuchi [3], Bando [2]) *Assume $\mathcal{E} \neq \emptyset$. Then*

- (i) *K -energy is bounded from below on \mathcal{K} and takes its absolute minimum exactly on \mathcal{E} .*
- (ii) *\mathcal{E} consists a single $Aut_0(M)$ orbit.*

Indeed the normalization constant $a(t)$ can be estimated by the K -energy.

Lemma 2.5. *Let φ be the solution of (2.3). The relation between $a(t)$ and the K -energy $\nu(\varphi)$ is given by*

$$(2.9) \quad a(t) + \nu(\varphi) = a(0) + \nu(\varphi_0).$$

Proof. We calculate the evolution of $a(t)$ along the Kähler-Ricci flow directly,

$$V \frac{d}{dt} a(t) = - \int_M (\Delta_\varphi + 1) \dot{\varphi} \omega_\varphi^n - \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + \varphi \right) \Delta_\varphi \dot{\varphi} \omega_\varphi^n.$$

According to the Stokes' theorem and (2.5) the first term vanishes identically. Meanwhile, by using the integration-by-part formula and (2.3), the second term becomes

$$\int_M (S_\varphi - n) \dot{\varphi} \omega_\varphi^n.$$

Since (2.6) implies

$$\frac{d}{dt} \nu(t) = - \frac{1}{V} \int_M (S_\varphi - n) \dot{\varphi} \omega_\varphi^n,$$

we obtain

$$(2.10) \quad \frac{d}{dt} a(t) = - \frac{d}{dt} \nu(t).$$

Thus, the assertion follows by integrating both sides of (2.10) with respect to t . \square

Since the K -energy is decreasing along the Kähler-Ricci flow, according to Theorem 2.4 we immediately conclude that:

Corollary 2.6. *On a Kähler-Einstein manifold, $a(t)$ is uniformly bounded along the Kähler-Ricci flow.*

The following theorems including the short time existence, the regularity and the continuous dependence on initial data of the Kähler-Ricci flow have been proved in Chen-Ding-Zheng [11], in which they defined a new second order Monge-Ampère flow called pseudo-Calabi flow

$$(2.11) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = -f(\varphi), \\ \Delta_{\varphi} f(\varphi) = S(\varphi) - \underline{S}. \end{cases}$$

The pseudo-Calabi flow coincides with the Kähler-Ricci flow, when the initial datum is restricted in the canonical Kähler class. Let $X = C^0([0, T], C^{2+\alpha}(M, g)) \cap C^1([0, T], C^{\alpha}(M, g))$.

Theorem 2.7. (Chen-Ding-Zheng [11]) *Let $\varphi_0 \in C^{2,\alpha}(M, g)$ be such that $\lambda\omega \leq \omega_{\varphi_0} \leq \Lambda\omega$ for two positive constants λ and Λ . Then the pseudo-Calabi flow has a unique solution $\varphi(x, t) \in X$, where T is the maximal existence time.*

Theorem 2.8. (Chen-Ding-Zheng [11]) *The solution of the pseudo-Calabi flow $\varphi \in X$ is smooth for any $t > 0$.*

Theorem 2.9. (Chen-Ding-Zheng [11]) *If ϕ is the solution of the pseudo-Calabi flow for initial datum ϕ_0 on $[0, T]$, then there is a neighborhood U of ϕ_0 such that the pseudo-Calabi flow has a solution $\varphi(t)$ on $[0, T]$ for any $\varphi_0 \in U$ and the mapping $\varphi_0 \mapsto \varphi(t)$ is C^k for $k = 0, 1, 2, \dots$*

A direct corollary of the continuous dependence on initial data Theorem 2.9 says,

Theorem 2.10. (Chen-Ding-Zheng [11]) *If M admits a cscK metric ω . Let $\varphi_0 \in C^{2,\alpha}(M, g)$ be such that $\lambda\omega \leq \omega_{\varphi_0} \leq \Lambda\omega$ for two positive constants λ and Λ . Then for any $T > 0$ there exists a positive constant $\epsilon_0(T)$. If $|\varphi_0|_{C^{2,\alpha}(M, g)} \leq \epsilon_0(T)$, then the pseudo-Calabi flow has a unique solution on $[0, T]$, and*

$$|\dot{\varphi}|_{C^{\alpha}(M, g)} + |\varphi|_{C^{2,\alpha}(M, g)} \leq C\epsilon_0(T)$$

for all $t \in [0, T]$, where C depends on M, g and T . Furthermore $\epsilon_0(T)$ goes to zero, as T goes to infinity.

3. NO NONTRIVIAL HOLOMORPHIC VECTOR FIELDS

Let $\eta(M)$ be the set composed of all holomorphic vector fields on M . Now, we start with the case $\eta(M) = \phi$. We shall prove the following proposition in this section.

Proposition 3.1. *Assume M admits a Kähler-Einstein metric ω and has no holomorphic vector fields. There exists a small positive constant ϵ_0 , suppose the initial datum satisfies*

$$|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0,$$

then the Kähler-Ricci flow g_{φ} converges smoothly to g .

Proof. We at first show that under the assumption of the proposition, the solution of (2.3) always stays in some small ϵ_1 -neighborhood of the zero function.

Lemma 3.2. *For any $\epsilon_1 > 0$, there exists a small positive constant ϵ_0 . If*

$$|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0,$$

then $|\varphi(t)|_{2,\alpha} \leq \epsilon_1$ for all $t \in [0, +\infty)$.

Proof. Suppose that the conclusion fails, then there must exist a sequence of initial datum φ_s^0 such that

$$|\varphi_s^0|_{C^{2,\alpha}} \leq \frac{1}{s}.$$

By virtue of Theorem 2.10, we get a sequence of solutions $\varphi_s(t)$ satisfying the flow equations (2.3) with $\varphi_s(0) = \varphi_s^0$. Let T_s be the first time such that

$$(3.1) \quad |\varphi_s(T_s)|_{C^{2,\alpha}} = \epsilon_1 \text{ and } |\varphi_s(t)|_{C^{2,\alpha}} < \epsilon_1$$

on $[0, T_s)$. According to Theorem 2.10 again, we have $T_s \geq T_1 > 0$. Moreover, we apply Theorem 2.8 to (2.3) on $[T_s - 2a, T_s]$ for fixed a such that $0 < a < \frac{T_s}{2} - \frac{T_1}{4}$, then we obtain the uniform higher order bound of the sequence of the solutions

$$|\varphi_s|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, a), \forall k \geq 0$$

on $[T_s - a, T_s]$. Consequently, there is a subsequence of $\phi_s = \varphi_s(T_s)$ converges smoothly to ϕ_∞ satisfying

$$(3.2) \quad |\phi_\infty|_{C^{2,\alpha}} = \epsilon_1.$$

It is obvious that g_{ϕ_∞} is still a Kähler metric. Since the K -energy is not only well defined for φ_s^0 by (2.7) but also decreasing along the Kähler-Ricci flow, Theorem 2.4 implies

$$0 \leq \nu_\omega(\phi_s) \leq \nu_\omega(\varphi_s(0)) \leq \frac{C}{s}.$$

By passing the limit we obtain

$$\lim_{s \rightarrow \infty} \nu_\omega(\varphi_s) = \nu_\omega(\varphi_\infty) = 0.$$

According to Theorem 2.4, we obtain g_{ϕ_∞} is a Kähler-Einstein metric. From Theorem 2.4 we deduce that ϕ_∞ must be a constant. Furthermore the normalization condition $I(\phi_\infty) = 0$ gives rise to $\phi_\infty = 0$ which contradicts to (3.2) and the lemma follows. \square

According to Theorem 2.8 and $|\varphi(t)|_{C^{2,\alpha}} \leq \epsilon_1$ uniformly, we have that $|\varphi(t)|_{C^k} \leq C_k$ for any $k \geq 3$ away from $t = 0$. It follows that there is a subsequence of any sequence t_i converges smoothly to a limit function φ_∞ . Moreover, since the K -energy has lower bound and it decays along the flow, ω_{φ_∞} must be a Kähler-Einstein metric. This together with Theorem 2.4 and the normalization condition implies that $\varphi_\infty = 0$. Because t_i is chosen randomly, we conclude the Kähler-Ricci flow converges smoothly to the original Kähler-Einstein metric. \square

4. M ADMITS NONTRIVIAL HOLOMORPHIC VECTOR FIELDS

4.1. Choice and estimate of holomorphic transformations. When M admits holomorphic vector fields, we need to find an appropriate holomorphic transformation. Set $\mathcal{E}_0 \subset \mathcal{H}_0$ be the space of Kähler potentials of Kähler-Einstein metrics.

Let $\sigma_t^* \omega$ be any curve with $\sigma_0 = id$ in \mathcal{E}_0 , the tangent vector at ω is $\frac{d}{dt} \sigma_t^*|_{t=0} \omega = L_X \omega$. Here $X = (\sigma_t)_*^{-1} \partial_t \sigma_t|_{t=0}$ is the real part of some holomorphic vector field. Since $C_1(M) > 0$ implies M is simply connected by Kobayashi [25], we obtain $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X$ for some function θ_X . Hence, that the dimension of the space of holomorphic vector fields is finite infers which of \mathcal{E}_0 is finite. Moreover, according to Mabuchi [28], we have \mathcal{E}_0 is also a totally geodesic submanifold of \mathcal{H}_0 . Then the point $\rho \in \mathcal{E}_0$ realizes the shortest distance between φ and \mathcal{E}_0 is uniquely determined. In fact, according to Theorem 2.4 we obtain a holomorphic diffeomorphism $\sigma \in$

$Aut_0(M)$ such that $\sigma^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\rho$. The following invariance of the K -energy under the holomorphic transformation is known in Mabuchi [27].

Lemma 4.1. $\nu(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) = \nu(\omega, \omega_\varphi) = \nu(\omega_\rho, \omega_\varphi)$.

Proof. Since ω and ω_ρ are both Kähler-Einstein metrics, we have that

$$\begin{aligned} \nu(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) &= \nu(\sigma^*\omega, \omega_\varphi) = \nu(\omega_\rho, \omega_\varphi) \\ &= \nu(\omega_\rho, \omega) + \nu(\omega, \omega_\varphi) = \nu(\omega, \omega_\varphi). \end{aligned}$$

The first inequality holds by Lemma (5.4.1) in [27] and the last one follows by Theorem (5.3) in [27]. \square

In [11] we prove some lemmas regarding to the metric geometry of the space of Kähler-Einstein metrics. The following two lemmas show that when metrics stay close to ω , their projection metrics are uniformly bounded.

Lemma 4.2. *There exists a positive constant ϵ , for any ρ satisfies $d(0, \rho) \leq \epsilon$, we have $|\rho|_{C^{3,\alpha}} \leq C_2\epsilon$.*

Proof. Since \mathcal{E}_0 is a finite dimension Riemannian symmetric space, the small ϵ neighborhood near $\rho = 0$ in this submanifold can be pulled back by the exponential map exp_0 to the tangent space $T_0(\mathcal{E}_0)$ at 0. Denote $\psi = exp_0^{-1}(\rho)$. Then the length from ψ to 0 is ϵ . We obtain the norm induced by the distance on $T_0(\mathcal{E}_0)$ is equivalent to the $C^{2,\alpha}$ norm, since all norms on a finite-dimensional vector space are equivalent. Then we have $|exp_0^{-1}(\rho)|_{C^{2,\alpha}}$ is bounded by $C_1\epsilon$. Furthermore, since the exponential map is a diffeomorphism in the ϵ neighborhood near $\rho = 0$, we obtain $|\rho|_{C^{2,\alpha}} \leq C_2\epsilon$ for some constant C_2 and this lemma follows by an appropriate choice of ϵ . \square

Remark 4.1. In fact, we can improve the above conclusion in Lemma 4.2 for C^k of fix $k \geq 0$, not only for $C^{3,\alpha}$ norm.

Lemma 4.3. *There exists a positive constant ϵ_1 . If $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$, then $|\rho|_{C^{3,\alpha}} \leq C_4$ and $|\sigma|_h \leq C_5$. Here h is the left invariant metric in $Aut(M)$.*

Proof. Choose a path $\gamma_t = t\varphi - I(t\varphi) \in \mathcal{H}_0$ for $0 \leq t \leq 1$. Denote $d(0, \varphi)$ the distance between 0 and φ . Then by using (2.1), we compute

$$\begin{aligned} d(0, \varphi) &\leq L(\gamma_t) = \int_0^1 \left(\int_M \left(\frac{\partial \gamma_t}{\partial t} \right)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &= \int_0^1 \left(\int_M (\varphi - \partial_t I(t\varphi))^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \leq C_3\epsilon_1 \end{aligned}$$

for $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$. Moreover, the choice of the ρ implies

$$d(0, \rho) \leq d(0, \varphi) + d(\varphi, \rho) \leq 2d(0, \varphi) \leq C_3\epsilon_1$$

by the triangle inequality. From Lemma 4.2, it follows that $|\rho|_{C^{3,\alpha}} \leq C_4 = C_2C_3\epsilon_1$. Furthermore, using Lemma 4.6 in Chen-Tian [14], we derive $|\sigma|_h \leq C_5$ and the lemma follows. \square

Remark 4.2. Alternatively the holomorphic transformation can be derived by minimizing $I - J$ functional in Bando-Mabuchi's work [3], that will be further discussed in the Section 8. They use this minimizer to prove the uniqueness of the Kähler-Einstein metric when the first Chern class is positive. The minimizer of $I - J$

functional is not unique in general, since the second variation of this functional is not strictly positive. However, we observe that when the potential is small enough, the minimizer is unique. Furthermore, we prove a similar estimate Corollary 8.2 to Lemma 4.3.

4.2. Long time existence and Cheeger-Gromov convergence. Set

$$\mathcal{S}(\epsilon_1, C(k, \epsilon_1)) = \{\varphi \mid |\varphi|_{C^{2,\alpha}} \leq \epsilon_1; |\varphi|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1)\}.$$

It is obvious that $0 \in \mathcal{S}$.

Lemma 4.4. *For any $\epsilon > 0$, There is a small positive constant ϵ depends on ϵ and \mathcal{S} such that for any $\varphi \in \mathcal{S}$, if $\nu_\omega(\varphi) \leq \epsilon$, then $|(\sigma^{-1})^*(\varphi - \rho)|_{C^{2,\alpha}} < \epsilon$.*

Proof. If the conclusion fails, we assume there exist a positive constant ϵ and a sequence of $\varphi_s \in \mathcal{S}$ satisfying

$$\nu_\omega(\varphi_s) \leq \frac{1}{s}$$

such that

$$(4.1) \quad |(\sigma_s^{-1})^*(\varphi_s - \rho_s)|_{C^{2,\alpha}} \geq \epsilon.$$

Since $\varphi_s \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1))$, we obtain a subsequence φ_{s_j} of φ_s converges to φ_∞ in smooth sense. Let $\hat{\varphi}_s = (\sigma_s^{-1})^*(\varphi_s - \rho_s)$. Lemma 4.3 gives

$$|\rho_s|_{C^{3,\alpha}} \leq C_4 \text{ and } |\sigma_s|_h \leq C_5$$

which implies there are subsequences (using the same notation) of ρ_{s_j} and σ_{s_j} by the Azela-Ascoli theorem and the Bolzano-Weierstrass theorem respectively such that

$$\begin{aligned} \rho_{s_j} &\rightarrow \rho_\infty \text{ in } C^{3,\beta} \text{ sense for any } \beta < \alpha \\ \text{and } \sigma_{s_j} &\rightarrow \sigma_\infty \text{ in the left invariant metric.} \end{aligned}$$

Then combining with Lemma 4.1 which implies that

$$\nu_\omega(\varphi_\infty) = \nu_\omega(\hat{\varphi}_\infty) = 0$$

we derive $\hat{\varphi}_{s_j}$ converges to $\hat{\varphi}_\infty = (\sigma_\infty^{-1})^*(\varphi_\infty - \rho_\infty) \in \mathcal{E}_0$ in $C^{3,\beta}$ and $\sigma_\infty^* \omega = \omega + \partial\bar{\partial}\rho_\infty$. Moreover, according to Theorem 2.4, we have $\hat{\varphi}_\infty, \varphi_\infty \in \mathcal{E}_0$. We claim that

$$d(\varphi_\infty, \rho_\infty) = 0.$$

Otherwise for some sufficient large N , when $s_j > N$, $d(\varphi_{s_j}, \rho_{s_j}) = d(\varphi_{s_j}, \mathcal{E}_0)$ has a strictly positive lower bound. Since it is shown the distance function is at least C^1 in Chen [10], we have $d(\varphi_\infty, \mathcal{E}_0) > 0$ that contradicts to $\varphi_\infty \in \mathcal{E}_0$. Consequently, this claim holds and implies $\hat{\varphi}_\infty = 0$ which is a contradiction to $|\hat{\varphi}_\infty|_{C^{2,\alpha}} \geq \epsilon$ given by (4.1). \square

Proposition 4.5. *Assume M admits a Kähler-Einstein metric ω and has nontrivial holomorphic vector fields. There is a small positive constant ϵ_0 . If $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, then there is a unique solution $\varphi(t)$ and the corresponding holomorphic transformation $\varrho(t)$ such that the normalization potential of $\varrho(t)^*\omega(t)$ always stays in \mathcal{S} . Moreover, for any sequence $\omega(t_i)$, there is a subsequence $\omega(t_{i_j})$ such that $\varrho(t_{i_j})^*\omega(t_{i_j})$ converges smoothly to a Kähler-Einstein metric ω_∞ .*

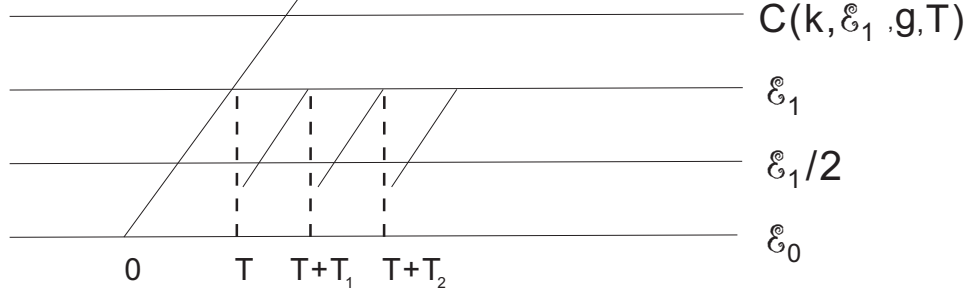


FIGURE 1. Illustration of the proof idea of Proposition 4.5 : solving the equation after pulling-back.

Proof. We prove this proposition by the contradiction method. Let ϵ_1 be determined in Lemma 4.3. Owing to Theorem 2.10, we assume there is a maximal time T such that

$$|\varphi|_{C^{2,\alpha}} < \epsilon_1 \text{ on } [0, T) \text{ and } |\varphi(T)|_{C^{2,\alpha}} = \epsilon_1.$$

According to Theorem 2.8 we obtain $|\varphi(T)|_{C^{k,\alpha}} \leq C(k, \epsilon_1, \frac{T}{2})$ on $[\frac{T}{2}, T]$. So we get

$$\varphi(T) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1, \frac{T}{2})).$$

There are two situations. If $\varphi(T)$ is a Kähler-Einstein metric, the flow will stop here and our theorem is proved. Otherwise, we will extend the flow as follows. We first choose ϵ_0 small enough to guarantee

$$\nu_\omega(\varphi_0) \leq o(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1, \frac{T}{2})))$$

where the constant $o(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1, \frac{T}{2})))$ is determined in Lemma 4.4. Let the holomorphic transformation σ be the projection of $\varphi(T)$ in \mathcal{E}_0 with $\sigma^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\rho$. We set φ_1^0 be the Kähler potential of the metric pulled back by σ , i.e.

$$(\sigma^{-1})^*\omega_{\varphi(T)} = \omega + \sqrt{-1}\partial\bar{\partial}[(\sigma^{-1})^*(\varphi(T) - \rho)] = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_1^0.$$

Since the K -energy is decreasing along the Kähler-Ricci flow, we obtain according to Lemma 4.1,

$$(4.2) \quad \nu_\omega(\varphi_1^0) \leq o(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1, \frac{T}{2}))).$$

So Lemma 4.4 implies that

$$(4.3) \quad |\psi|_{C^{2,\alpha}(g)} = |(\sigma^{-1})^*(\varphi(T) - \rho)|_{C^{2,\alpha}(g)} < \frac{\epsilon_1}{2}.$$

We next show that the Kähler-Ricci flow is invariant under the transformation. Let $\varphi_1 = (\sigma^{-1})^*(\varphi(t) - \rho)$. Since

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_1 &= (\sigma^{-1})^*[\log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \frac{1}{V} \int_M (\log \frac{\omega_\varphi^n}{\omega^n} + \varphi)\omega_\varphi^n] \\ &= (\sigma^{-1})^*[\log \frac{\omega_\varphi^n}{\omega_\rho^n} + \varphi - \rho - \frac{1}{V} \int_M (\log \frac{\omega_\varphi^n}{\omega_\rho^n} + \varphi - \rho)\omega_\varphi^n] \\ &= \log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1 - \frac{1}{V} \int_M (\log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1)\omega_{\varphi_1}^n \end{aligned}$$

The second equality follows from the fact that ω_ρ is a Kähler-Einstein metric. We conclude that φ_1 is the solution of the equation of the form

$$(4.4) \quad \begin{cases} \frac{\partial}{\partial t} \varphi_1 &= \log \frac{\omega_{\varphi_1}^n}{\omega_\rho^n} + \varphi_1 + a(t) \\ \varphi_1(0) &= \varphi_1^0 = (\sigma^{-1})^*(\varphi(T) - \rho) \end{cases}$$

with (4.3) and (4.2). Again, Theorem 2.10 implies (4.4) has a solution on $[0, T_1]$ with $T_1 \geq T$ such that

$$|\varphi_1(T_1)|_{C^{2,\alpha}} = \epsilon_1.$$

Moreover, let $\varphi(t) = \sigma^* \varphi_1(t - T) + \rho$ on $[T, T + T_1]$, the new $\varphi(t)$ is the solution of (2.3) on $[0, T + T_1]$. Then we repeat the same steps inductively for

$$\varphi_{s-1}(T_{s-1}) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1, \frac{T}{2})),$$

with $T_{s-1} \geq T$ which is obtained in Theorem 2.10 till φ_s becomes a Kähler-Einstein at time T_s , if $T_s < \infty$. If not, we have the Kähler-Ricci flow has the long time existence and the solution $\varphi(t)$ for all $t \geq 0$ given by

$$\omega_{\varphi(t)} = \prod_{i=0}^{s-1} \sigma_i^* \omega_{\varphi_s(t)} \text{ on } [\sum_{i=0}^{s-1} T_i, \sum_{i=0}^s T_i).$$

For any sequence $\{\varphi_{t_j}\}$, there is s such that $\sum_{i=0}^{s-1} T_i \leq t_j \leq \sum_{i=0}^s T_i$. Furthermore, let $\varrho_j = (\prod_{i=0}^{s-1} \sigma_i)^{-1}$. We have

$$|\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^\alpha} \leq \epsilon_1 \text{ and } |\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^k} \leq C(k, \epsilon_1, \frac{T}{2}).$$

Therefore all metrics are equivalent and their derivatives are bounded. We denote

$$\omega_{\psi_{t_j}} = \varrho_j^* \omega_{\varphi_{t_j}}.$$

It follows that by abuse of notation there is a subsequence of $\omega_{\psi_{t_j}}$ converges to a limit metric ω_∞ . However, ω_∞ depends on the choice of the subsequence. Since the K -energy is bounded below, we have $\lim_{s \rightarrow \infty} \nu(\omega, \omega_{\psi_{t_j}}) = 0$. It follows that g_∞ is a Kähler-Einstein metric from Theorem 2.4. Consequently, this proposition is proved. \square

Let $t_s = \sum_{i=0}^s T_i$. Follow the same argument in Chen-Tian [14], we can first connect each disperse points by geodesics in the space of Kähler-Einstein metric so that

$$\varrho(t) = \varrho(s) \exp((t - s)X_s), \forall t \in [s, s + 1]$$

for X_s is uniformly bounded by Lemma 4.3. Then smooth the corner at each points t_s by replacing the broken line by a smooth curve in small neighborhood near t_s without changing the value and t derivative in the end points. Hence we have extended the holomorphic transformation to each t so that it is Lipschitz continuous in t .

5. EXPONENTIAL CONVERGENCE

In this section, we show that the sequence of holomorphic transformations $\varrho(t)$ are compact and the exponential convergence of the Kähler-Ricci flow. Let $\omega_{\psi(t)} = \varrho(t)^*\omega_{\varphi(t)}$. We have already obtained $\lim_{t \rightarrow \infty} Ric(g_{\psi_t}) - \omega_{\psi_t} = 0$ in Section 4. Since the holomorphic transformation keeps this identity invariant, we obtain

$$(5.1) \quad \lim_{t \rightarrow \infty} Ric(g_{\varphi_t}) - \omega_{\varphi_t} = 0.$$

After taking the trace, we get

$$(5.2) \quad \lim_{t \rightarrow \infty} \Delta_{\varphi_t} \dot{\varphi} = \lim_{t \rightarrow \infty} S(g_{\varphi_t}) - n = 0.$$

Now for each time t , we apply the De Giorgi iteration to derive the L^∞ bound of $\dot{\varphi}$ under the normalization condition (2.5). In the following, the constant C may be different from line to line.

Lemma 5.1. *Along the Kähler-Ricci flow, we have*

$$(5.3) \quad \|\dot{\varphi}\|_\infty \leq C\|S - n\|_\infty \text{ for any } t > 0.$$

Proof. We first notice that $\dot{\varphi}$ satisfies the equation

$$\Delta_{\varphi} \dot{\varphi} = -S(g_{\varphi_t}) + n.$$

Then we multiply this equation with $\dot{\varphi}^{2q-1}$ and integrate on M to get

$$(5.4) \quad \frac{2q-1}{q^2} \int_M |\nabla(\dot{\varphi}^q)|^2 \omega_\varphi^n = \int_M \dot{\varphi}^{2q-1} (S - n) \omega_\varphi^n.$$

Since $\omega_\varphi = \sigma^* \omega_\psi$ and the Sobolev constant and Poincare constant are invariant under the holomorphic transformation, we have they are uniformly bounded. Let $q = 1$, we have

$$(5.5) \quad \int_M |\nabla \dot{\varphi}|^2 \omega_\varphi^n = \int_M \dot{\varphi} (S - n) \omega_\varphi^n.$$

We apply the Poincare inequality and (2.5) to (5.5) to get

$$\|\dot{\varphi}\|_{L^2}^2 \leq C_p^2 \int_M \dot{\varphi} (S - n) \omega_\varphi^n.$$

Here C_p is the Poincare constant. Then we use the $\frac{1}{2}$ -Hölder inequality to get

$$(5.6) \quad \begin{aligned} \|\dot{\varphi}\|_{L^2}^2 &\leq \frac{1}{2} \|\dot{\varphi}\|_{L^2}^2 + \frac{1}{2} C_p^4 \|S - n\|_\infty^2 \\ &\leq C_p^4 C(\delta) \|S - n\|_\infty^2 \end{aligned}$$

and apply the $\frac{1}{2}$ -Hölder inequality to (5.5) again to get

$$\int_M |\nabla \dot{\varphi}|^2 \omega_\varphi^n \leq \|\dot{\varphi}\|_{L^2}^2 + \|S - n\|_\infty^2.$$

Accordingly, combining with (5.6) we obtain

$$(5.7) \quad \|\dot{\varphi}\|_{W^{1,2}} \leq C\|S - n\|_\infty.$$

Let $p > n$. We claim:

$$(5.8) \quad \|\dot{\varphi}\|_{\frac{2np}{2n+p}} \leq C\|S - n\|_\infty.$$

When $n = 1$, the claim obviously follows from (5.7) by the Sobolev imbedding theorem. Since the Sobolev constant is bounded, we can use the Sobolev inequality

$$(5.9) \quad C_1 \left(\int_M |f|^{\frac{2n}{n-1}} \omega_\varphi^n \right)^{\frac{n-1}{n}} \leq C_2 \int_M |f|^2 \omega_\varphi^n + \int_M |\nabla f|^2 \omega_\varphi^n.$$

For any $n \geq 2$, we apply the Sobolev inequality (5.9) to the left side of (5.4) and the Hölder inequality to the right one to obtain

$$\left(\int_M |\dot{\varphi}|^{\frac{2nq}{n-1}} \omega_\varphi^n \right)^{\frac{n-1}{n}} \leq C \int_M |\dot{\varphi}|^{2q} \omega_\varphi^n + \int_M (S - n)^{2q} \omega_\varphi^n.$$

Setting $q = \frac{(n-1)p}{2n+p}$, we have

$$(5.10) \quad \left(\int_M |\dot{\varphi}|^{\frac{2np}{2n+p}} \omega_\varphi^n \right)^{\frac{n-1}{n}} \leq C \int_M |\dot{\varphi}|^{\frac{2(n-1)p}{2n+p}} \omega_\varphi^n + \int_M (S - n)^{\frac{2(n-1)p}{2n+p}} \omega_\varphi^n.$$

Since the L^p interpolation inequality implies

$$\|\dot{\varphi}\|_{\frac{2(n-1)p}{2n+p}} \leq \epsilon \|\dot{\varphi}\|_{\frac{2np}{2n+p}} + C(\epsilon) \|\dot{\varphi}\|_2,$$

we have by (5.7)

$$(5.11) \quad \|\dot{\varphi}\|_{\frac{2(n-1)p}{2n+p}} \leq \epsilon \|\dot{\varphi}\|_{\frac{2np}{2n+p}} + C(\epsilon) \|S - n\|_\infty.$$

Hence substituting (5.11) into (5.10) we obtain the claim provided $C\epsilon = \frac{1}{2}$.

Next, let x be any point in M and η be a smooth cut-off function defined in the closed ball $B_{2r}(x)$ centered at the point x such that η equals to 1 within $B_{\frac{1}{2}r}(x)$ and vanishes outside $B_r(x)$. In $B_{2r}(x)$, by chain and product rules $\eta\dot{\varphi}$ satisfies

$$(5.12) \quad \begin{aligned} \Delta_\varphi(\eta\dot{\varphi}) &= (\eta - 2\Delta\eta)(-S(g_{\varphi_t}) + n) + \Delta_\varphi\eta\dot{\varphi} + 2\nabla^i(\nabla_i\eta(-S(g_{\varphi_t}) + n)) \\ &\doteq f + \nabla^i g_i \end{aligned}$$

for $f = (\eta - 2\Delta\eta)(-S(g_{\varphi_t}) + n) + \Delta_\varphi\eta\dot{\varphi}$ and $g = 2(\nabla_i\eta(-S(g_{\varphi_t}) + n))$. Set $F = (\|f\|_{\frac{2np}{2n+p}} + \|g_i\|_p)$ and the test function ψ be $\max\{k, \eta\dot{\varphi}\}$ for any fixed $k > 0$. Multiplying (5.12) with ψ and integrating by part on M , we have

$$(5.13) \quad \int_M |\nabla\psi|^2 \omega_\varphi^n = - \int_M f\psi \omega_\varphi^n + \int_M g_i \psi^i \omega_\varphi^n.$$

By using the Hölder inequality to the right side of (5.13) we obtain

$$\|\nabla\psi\|_2^2 \leq F(\|\nabla\psi\|_2 + \|\psi\|_{\frac{2n}{n-1}}) |A(k)|^{\frac{1}{2} - \frac{1}{p}}$$

for $A(k) = \{y \in B_r(x) | (\eta\dot{\varphi})(y) > k\}$. Using the Sobolev inequality (5.9) to ψ , we have

$$(5.14) \quad \begin{aligned} C_1 \left(\int_M |\psi|^{\frac{2n}{n-1}} \omega_\varphi^n \right)^{\frac{n-1}{n}} &\leq C_2 \int_M |\psi|^2 \omega_\varphi^n + \int_M |\nabla\psi|^2 \omega_\varphi^n \\ &\leq C_2 |B_r|^{\frac{1}{n}} \left(\int_M |\psi|^{\frac{2n}{n-1}} \omega_\varphi^n \right)^{\frac{n-1}{n}} + \int_M |\nabla\psi|^2 \omega_\varphi^n \\ &\leq 2 \int_M |\nabla\psi|^2 \omega_\varphi^n \end{aligned}$$

when we choose r small enough so that $C_2 |B_r|^{\frac{1}{n}} \leq \frac{C_1}{2}$. Furthermore, using (5.14) we derive

$$\|\nabla\psi\|_2^2 \leq CF \|\nabla\psi\|_2 |A(k)|^{\frac{1}{2} - \frac{1}{p}}.$$

Then we apply the Hölder inequality to obtain

$$\begin{aligned} \|\nabla\psi\|_2^2 &\leq \frac{1}{2}\|\nabla\psi\|_2^2 + \frac{1}{2}C^2F^2|A(k)|^{1-\frac{2}{p}} \\ &\leq C^2F^2|A(k)|^{1-\frac{2}{p}}. \end{aligned}$$

Now using (5.14) again we derive

$$\|\psi\|_{\frac{2n}{n-1}} \leq CF|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

which provides for any $h > k$,

$$|A(h)| \leq \left(\frac{CF}{h-k}\right)^{\frac{2n}{n-1}} |A(k)|^{\frac{n(p-2)}{p(n-1)}}.$$

Let $d = CF|A(0)|^{\frac{p-2n}{2np}} 2^{\frac{n(p-2)}{p-2n}}$ and $k_s = d(1 - \frac{1}{2^s})$. It is proved inductively that

$$|A(k_{s+1})| \leq |A(0)| 2^{-\frac{2nps}{p-2n}}.$$

Consequently, $|A(d)| = 0$ as $s \rightarrow \infty$. In other words, we obtain

$$\dot{\varphi} \leq CV^{\frac{p-2n}{2np}} 2^{\frac{n(p-2)}{p-2n}} (\|\dot{\varphi}\|_{\frac{2np}{2n+p}} + \|S - n\|_\infty) \text{ in } B_{\frac{1}{2}r}(x).$$

Finally, this estimate together with (5.8) provides the global supper bound of $\dot{\varphi}$ we desired. Similarly, we obtain the global lower bound. \square

Remark 5.1. Given the bound of the scalar curvature (5.2) and the K -energy's lower bound, it is available to apply Phong-Sturm's argument [32] to get $\int_M |\nabla\dot{\varphi}|^2 \omega_\varphi^n \rightarrow 0$. Then we apply the Poincare inequality and the normalization condition (2.5) to obtain $\int_M |\dot{\varphi}|^2 \omega_\varphi^n \rightarrow 0$.

The evolution of $\mu_0(t) = \frac{1}{V} \int_M \dot{\varphi}^2 \omega_\varphi^n$ is given by

$$\frac{\partial}{\partial t} \mu_0(t) = -2 \int_M (1 + \dot{\varphi}) |\nabla\dot{\varphi}|^2 \omega_\varphi^n + 2 \int_M \dot{\varphi}^2 \omega_\varphi^n.$$

Then following the same argument in Chen-Tian [13], we use the fact the the spectrum of g_t converges to the spectrum of the Kähler-Einstein metric and the Futaki invariant is zero to obtain

$$(5.15) \quad \mu_0(t) = \frac{1}{V} \int_M \dot{\varphi}^2 \omega_\varphi^n \leq \mu_0(0) e^{-\theta t}.$$

Moreover it is direct to compute that (see for Page 539 in Chen-Tian [13])

$$(5.16) \quad \mu_l(t) = \frac{1}{V} \int_M |\nabla^l \dot{\varphi}|^2 \omega_\varphi^n \leq \mu_l(0) e^{-\theta t}.$$

We apply the Sobolev imbedding theorem to obtain

$$(5.17) \quad |\dot{\varphi}|_{C^l(g_\varphi)} \leq C e^{-\theta t}.$$

We use $\varphi(t) = \varphi(0) + \int_0^1 \dot{\varphi} dt$ and (5.17) to obtain the C^0 estimate

$$(5.18) \quad |\varphi(t)|_{C^0} \leq |\varphi(0)|_{C^0} + C e^{-\theta t}.$$

From the equation $\dot{\varphi} - \log \frac{\omega_\varphi^n}{\omega^n} = \varphi + a(t)$, Yau's estimate [42] (see [7]) gives the second order estimate $0 < n + \Delta\varphi \leq C$. It follows g and g_φ are equivalent. So we have $|\dot{\varphi} - \varphi - a(t)|_{C^2} \leq C$ thanks to the uniform bound of $a(t)$ given by (2.9) and (5.17). Then the $C^{2,\alpha}$ estimates by Evans [19] and Krylov [26] shows φ has

uniform $C^{2,\alpha}$ bound. Moreover, Theorem 2.8 provides the uniform bound on all higher order derivatives. Then g and g_ϕ are C^∞ equivalent. So (5.16) implies

$$|\varphi - \varphi_\infty|_{C^k(g_\infty)} \leq C_k e^{-\theta t}, \forall k \geq 0.$$

Therefore, we have obtained the exponential convergence of the Kähler-Ricci flow.

Proposition 5.2. *If the Kähler-Ricci flow converges to a Kähler-Einstein metric in Cheeger-Gromov sense, i.e. for any sequence $g(t_i)$, there is a subsequence $g(t_{i_j})$ and the holomorphic transformation $\varrho(t_{i_j})$ such that $\varrho(t_{i_j})^*g(t_{i_j})$ converges smoothly to a Kähler-Einstein metric g_∞ . Then the Kähler-Ricci flow must converge exponentially to a unique Kähler-Einstein metric nearby.*

6. KÄHLER-RICCI SOLITON

In this section we generalize our above argument to the Kähler-Ricci solitons. According to Fujiki [20], The identity part of holomorphic transformation group $Aut_0(M)$ is meromorphically isomorphic to a linear algebraic group $L(M)$ and such that the quotient $Aut_0(M)/L(M)$ is a complex torus. In Futaki-Mabuchi's work [22], they used the Chevalley decomposition to $L(M)$ to obtain a semidirect decomposition

$$Aut_0(M) = Aut_r(M) \ltimes R_u.$$

Here $Aut_r(M)$ is the reductive algebra group which is the complexification of a maximal compact subgroup K and R_u is the unipotent radical of $Aut_0(M)$. Let η_r be the Lie algebra of $Aut_r(M)$. A Kähler metric ω is called Kähler-Ricci soliton, if there is a holomorphic vector field X such that

$$L_X \omega = Ric - \omega.$$

Tian-Zhu in [38] proved the uniqueness of Kähler-Ricci soliton for a fixed X in the Lie algebra of $Aut_0(M)$:

Theorem 6.1. *(Tian-Zhu [38]) If (ω, X) and (ω', X) are two Kähler-Ricci solitons, then there are holomorphic transformation groups $\sigma \in Aut_0(M)$ and $\tau \in Aut_r(M)$ such that $\sigma^*\omega = \tau^*\sigma^*\omega'$ and $\sigma^*X \in \eta_r$.*

Without loss of generality, we may assume $X \in \eta_r(M)$. Since $L_{\Im X} \omega = 0$, $\Im X$ generates a one-parameter isometric group K_X . We further choose K such that $K_X \subseteq K$. According to Proposition 2.1 in Tian-Zhu [39], X lies in the center of η_r .

Since there is a real value function θ_X such that $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X$ with $\int_M e^{\theta_X} \omega^n = V$ by the Hodge theory. Then the potential equation of the Kähler-Ricci flow (1.2) is

$$(6.1) \quad \begin{cases} \frac{\partial \varphi}{\partial t} &= \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \theta_X + a(t) \\ \varphi(0) &= \varphi_0. \end{cases}$$

We choose

$$a(t) = -\frac{1}{V} \int_M (\log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \theta_X) \omega_\varphi^n$$

and $I(\varphi_0) = 0$ so that the Kähler-Ricci flow stays in \mathcal{H}_0 . Perelman in [31] defined a functional called W functional,

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(|\nabla f|^2 + S) + f - n] e^{-f} dV$$

which is invariant under diffeomorphism σ and scaling C : $\mathcal{W}(C\sigma^*g, \sigma^*f, C\tau) = \mathcal{W}(g, f, \tau)$. And the μ functional is defined by

$$(6.2) \quad \mu(g, \tau) = (4\pi\tau)^{-\frac{n}{2}} \inf_{\int_M e^{-f} dV = 1} \mathcal{W}(g, f, \tau)$$

which is also invariant under diffeomorphism. The minimum is achieved by some smooth function f satisfying $\tau[(2\Delta f - |\nabla f|^2) + S] + f - n = \mu(g, \tau)$. The first variation of $\mu(g, \tau)$ at $g'_{ij} = v_{ij}$ for fixed τ is

$$\mu'(v_{ij}, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M \left\{ -\tau(v_{ij}, Ric + D^2 f - \frac{1}{2\tau}g) \right\} e^{-f} dV_g.$$

So the (shrinking) Kähler-Ricci soliton is the critical point of $\mu(g, \tau = \frac{1}{2})$. The gradient flow of the μ functional equals to (1.1) with $\lambda = 1$ up to a diffeomorphism generated by ∇f . So the μ functional is nondecreasing along the Ricci flow. Tian-Zhu (see Proposition 2.1 in Tian-Zhu [41]) calculated the second variant of this functional near a Kähler-Ricci soliton in the canonical class.

Theorem 6.2. (*Tian-Zhu [41]*) *It holds*

$$(6.3) \quad \frac{\partial^2}{\partial t^2} \mu(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)|_{t=0} \leq 0$$

and the equality holds if and only if $\dot{\varphi}(0)$ is the real part of the holomorphic potential of some holomorphic vector field.

So the only directions at a Kähler-Ricci soliton ω in (6.3) vanishes are the the directions tangent to the orbit of ω under the action of $Aut_0(M)$ and we obtain the following local property of the μ functional.

Lemma 6.3. *Kähler-Ricci soliton is the local maximum of $\mu(g)$ in the canonical class.*

As a result we deduce that a Kähler metric which achieves the maximum value of the $\mu(g)$ functional near a Kähler-Ricci soliton must be a Kähler-Ricci soliton.

Let $\mathcal{E}_0 \subset \mathcal{H}_0$ be the space of potentials of Kähler-Einstein solitons with respect to the holomorphic vector field X . In fact, due to Theorem 6.1, \mathcal{E}_0 is a single orbit under the action of $Aut_r(M)$. Moreover, analogously to the extremal metric in Calabi [5], in the appendix of Tian-Zhu [38], their Lemma A.2. and Theorem A shows that the identity component of the holomorphic isometric group of the Kähler-Ricci soliton (ω, X) is a maximal compact subgroup of $Aut_r(M)$ containing K_X . So $(Aut_r(M), K)$ is a Riemannian symmetric pair and \mathcal{E}_0 is $Aut_r(M)$ -equivariantly diffeomorphic to the Riemannian symmetric space $Aut_r(M)/K$. Then each geodesic initials from ω in \mathcal{E}_0 is written in the form $\mathfrak{g}(t) = \exp(t\Re Y)^*\omega$ for some nonzero Y whose imagine part is a Killing vector field. According to Theorem 3.5 in Mabuchi [28], we obtain $\mathfrak{g}(t)$ is also a geodesic in \mathcal{K} . Thus we obtain:

Lemma 6.4. *\mathcal{E}_0 is a finite dimension totally geodesic submanifold of \mathcal{H}_0 .*

If we choose $\omega_\rho = \omega + \partial\bar{\partial}\rho$ such that ρ realizes the shortest distance between ψ and \mathcal{E}_0 . Clearly, ρ is uniquely determined. In fact, due to Theorem 6.1 we obtain a holomorphic diffeomorphism $\sigma \in Aut_r(M)$ such that $\sigma^*\omega = \omega_\rho = \omega + \sqrt{-1}\partial\bar{\partial}\rho$ and $\rho \in \mathcal{E}_0$. Following the analogous argument of Proposition 4.5 by using the μ functional instead of the K -energy, we obtain the following proposition.

Proposition 6.5. *Assume M admits a Kähler-Ricci soliton (ω, X) . There exists a small constant ϵ_0 . If $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, then there is a unique solution $\varphi(t)$ and the corresponding holomorphic transformation $\varrho(t) \in \text{Aut}_r(M)$ such that normalization potential of $\varrho(t)^*\omega_\varphi(t)$ always stays in \mathcal{S} . Moreover, for any sequence $g(t_i)$, there is a subsequence $g(t_{i_j})$ such that $\varrho(t_{i_j})^*g_{\varphi(t_{i_j})}$ converges smoothly to g_∞ .*

Let ς be generated by $\Re X$ such that $\Re X = (\varsigma^{-1})_* \frac{\partial}{\partial t} \varsigma$, $\varsigma^* \omega = \omega_\varrho$ and $\phi = \varsigma^* \varphi + \varrho$. We obtain the modified Kähler-Ricci flow of the form

$$(6.4) \quad \begin{cases} \frac{\partial}{\partial t} \omega_\phi &= -\text{Ric}(\omega_\phi) + \omega_\phi + L_{\Re X} \omega_\phi, \\ \omega_{\phi(0)} &= \omega_{\varphi_0}. \end{cases}$$

The modified potential equation is

$$(6.5) \quad \begin{cases} \frac{\partial \phi}{\partial t} &= \log \frac{\omega_\phi^n}{\omega^n} + \phi + \Re X(\phi) + a(t), \\ \phi(0) &= \varphi_0, \end{cases}$$

with the normalization condition

$$(6.6) \quad a(t) = -\frac{1}{V} \int_M (\log \frac{\omega_\phi^n}{\omega^n} + \phi + \Re X(\phi)) \omega_\phi^n.$$

Here

$$\begin{aligned} L_X \omega_\phi &= \text{Ric} - \omega + \partial \bar{\partial}(X(\phi)) = \partial \bar{\partial} \theta_X(\phi) = \partial \bar{\partial}(\theta_X + X(\phi)), \\ L_{\Re X} \omega_\phi &= \partial \bar{\partial}(\theta_X + \Re X(\phi)) \text{ and } L_{\Im X} \omega_\phi = \partial \bar{\partial}(\Im X(\phi)). \end{aligned}$$

When the initial datum is K_X -invariant, φ and ϕ are both K_X -invariant. Choose the K_X -invariant Kähler potential space be

$$M_X(\omega) = \{\phi \in C^\infty(M) \mid \omega + \partial \bar{\partial} \phi > 0, \Im X(\phi) = 0\}.$$

In Tian-Zhu [39], they introduced the modified Futaki invariant

$$F_X(Y) = \int_M Y(f - \theta_X) e^{\theta_X} \omega^n$$

for all $X, Y \in \eta(M)$ and the Futaki potential f determined by ω and the modified K -energy

$$\begin{aligned} \mu(\omega, \omega_\phi) &= -\frac{n}{V} \int_0^1 \int_M \dot{\phi} [\text{Ric}(\omega_\phi) - \omega_\phi - \partial \bar{\partial} \theta_X(\phi) \\ &\quad + \partial(h_{\omega_\phi} - \theta_X(\phi)) \wedge \bar{\partial} \theta_X(\phi)] e^{\theta_X(\phi)} \omega_\phi^{n-1} dt. \end{aligned}$$

for any ϕ in $M_X(\omega)$. Along the modified Kähler-Ricci flow (6.4), the modified Futaki invariant takes the form

$$(6.7) \quad \begin{aligned} F_X(Y) &= - \int_M Y(\dot{\phi}) e^{\theta_X(\phi)} \omega_\phi^n \\ &= - \int_M \theta_Y(\Delta + X) \dot{\phi} e^{\theta_X(\phi)} \omega_\phi^n. \end{aligned}$$

It is obvious to obtain the evolution of the modified K -energy along the modified Kähler-Ricci flow, i.e.

$$(6.8) \quad \frac{\partial}{\partial t} \mu(\omega, \omega_\phi) = -\frac{1}{V} \int_M |\nabla \dot{\phi}|^2 e^{\theta_X(\phi)} \omega_\phi^n.$$

Accordingly, the modified K -energy is decreasing along the modified Kähler-Ricci flow on $M_X(\omega)$.

Theorem 6.6. (Tian-Zhu [39]) *If M admits a Kähler-Ricci soliton (ω, X) , the modified Futaki invariant $F_X(Y) = 0$ for all $Y \in \eta(M)$, and the modified K -energy has lower bound on $M_X(\omega)$.*

By proposition 6.5, we have $\omega_{\psi}(t) = \varrho^* \omega_{\varphi}(t)$ converges to a Kähler-Ricci soliton,

$$\lim_{t \rightarrow \infty} Ric(g_{\psi_t}) - \omega_{\psi_t} - L_X \omega_{\psi_t} = 0.$$

Moreover, since X stays in the center of $\eta_r(M)$, we have the holomorphic transformation $\varsigma^* \varrho^{-1*}$ keeps the identity invariant,

$$(6.9) \quad \lim_{t \rightarrow \infty} Ric(g_{\phi_t}) - \omega_{\phi_t} - L_X \omega_{\phi_t} = 0.$$

After taking trace, we obtain

$$(6.10) \quad \lim_{t \rightarrow \infty} \Delta_{\phi_t} \dot{\phi} = \lim_{t \rightarrow \infty} S(g_{\phi_t}) - n - \Delta_{\phi_t} \theta_X(\phi) = 0.$$

Again since X stays in the center of $\eta_r(M)$, we also get

$$\lim_{t \rightarrow \infty} L_X(\sqrt{-1} \partial \bar{\partial} \dot{\phi}) = \lim_{t \rightarrow \infty} L_X(Ric(g_{\phi_t}) - \omega_{\phi_t} - L_X \omega_{\phi_t}) = 0.$$

For each time t , similarity to Lemma 5.1 we apply the De Giorgi iteration to deduce the L^∞ bound of $\dot{\phi}$ and $X(\dot{\phi})$. In fact, we obtain

$$(6.11) \quad \|\dot{\phi}\|_{W^{1,2}} \leq C \|S - n - \Delta \theta_X(\phi)\|_\infty.$$

Then we derive

$$(6.12) \quad \|\dot{\phi}\|_\infty \leq C \|S - n - \Delta \theta_X(\phi)\|_\infty$$

and the L^∞ bound of $X(\dot{\phi}) - \frac{1}{V} \int_M X(\dot{\phi}) \omega_\phi^n$

$$(6.13) \quad \|X(\dot{\phi}) - \frac{1}{V} \int_M X(\dot{\phi}) \omega_\phi^n\|_\infty \leq C \|tr_{g_\phi} L_X(Ric(g_{\phi_t}) - \omega_{\phi_t} - L_X \omega_{\phi_t})\|_\infty.$$

Moreover, Zhu [43] provides the following estimate.

Theorem 6.7. (Zhu [43]) *For any $\phi \in M_X(\omega)$ there is a constant C depends on g and X such that*

$$(6.14) \quad \|X(\phi)\|_\infty \leq C.$$

Lemma 6.8. *The relation between $a(t)$ and the modified K -energy is*

$$(6.15) \quad |a(t) - a(0)| \leq \mu(\omega, \omega_{\phi_0}) - \mu(\omega, \omega_{\phi_t}).$$

In addition,

$$(6.16) \quad |a'(t)| \leq C \frac{1}{V} \int_M |\nabla \dot{\phi}|^2 e^{\theta_X(\phi)} \omega_\phi^n \leq C \|S - n - \Delta \theta_X(\phi)\|_\infty.$$

Proof. We compute (6.6) to see that

$$(6.17) \quad a'(t) = -\frac{1}{V} \int_M [X(\dot{\phi}) - |\nabla \dot{\phi}|^2] \omega_\phi^n.$$

It is obvious that

$$(6.18) \quad \left| \frac{1}{V} \int_M X(\dot{\phi}) \omega_\phi^n \right| \leq C \frac{1}{V} \int_M |\nabla \dot{\phi}|^2 \omega_\phi^n.$$

Since $\theta_X(\phi) = \theta_X + X(\phi)$, then (6.16) follows from (6.14) and (6.11). Hence, integrating with t in both sides of (6.17) and using (6.8) we have (6.15). \square

So Theorem 6.6 implies $a(t)$ is uniformly bounded. Moreover, the L^∞ bound of $X(\dot{\phi})$ is obtained by substituting (6.18) and (6.11) in (6.13),

(6.19)

$$\|X(\dot{\phi})\|_\infty \leq C\|tr_{g_\phi} L_X(Ric(g_{\phi_t}) - \omega_{\phi_t} - L_X \omega_{\phi_t})\|_\infty + C\|S - n - \Delta\theta_X(\phi)\|_\infty.$$

It is direct to compute the t derivative of $v_0(t) = \frac{1}{V} \int_M \dot{\phi}^2 e^{\theta_X(\phi) - \dot{\phi}} \omega_\phi^n$, we arrive at

$$\frac{\partial}{\partial t} v_0(t) = \frac{1}{V} \int_M [-2(1 - \dot{\phi})|\nabla \dot{\phi}|^2 + \dot{\phi}^2(2 - \dot{\phi} - a') + 2\dot{\phi}a'] e^{\theta_X(\phi) - \dot{\phi}} \omega_\phi^n.$$

Hence we obtain for $\delta(t) = \|\dot{\phi}\|_\infty$,

$$\begin{aligned} \frac{\partial}{\partial t} v_0(t) &\leq \frac{1}{V} \int_M [-2(1 - \delta)e^{-\delta}|\nabla \dot{\phi}|^2 + \dot{\phi}^2(2 + \delta - a')e^\delta] e^{\theta_X(\phi)} \omega_\phi^n \\ &\quad + \frac{1}{V} 2a'e^\delta \int_M |\dot{\phi}|^2 e^{\theta_X(\phi)} \omega_\phi^n. \end{aligned}$$

In which $\delta(t) \rightarrow 0$ and $a'(t) \rightarrow 0$ follow from (6.12) and (6.16) respectively.

As it is known in Futaki [21] and Tian-Zhu [8], the operator $\Delta_g + X$ for a Kähler-Ricci soliton (g, X) is a self-adjoint elliptic operator in the space $C^\infty(M, \mathbb{C})$ equipped with the weighted inner product $(f, g) = \int_M f \bar{g} e^{\theta_X(g)} \omega^n$. Also the first eigenvalue of $\Delta_g + X$ is 1 and moreover the corresponding eigenspace consists the holomorphic potentials of the holomorphic vector fields in $\eta(M)$. We apply the same method to the Kähler-Einstein metric case to use the fact that the spectrum of g_ϕ converges to the spectrum of the Kähler-Ricci soliton and the modified Futaki invariant (6.7) vanishes to obtain

$$(6.20) \quad u_0(t) = \frac{1}{V} \int_M \dot{\phi}^2 \omega_\phi^n \leq u_0(0) e^{-\theta t}.$$

Moreover we show that

Lemma 6.9.

$$(6.21) \quad u_l(t) = \frac{1}{V} \int_M |\nabla^l \dot{\phi}|^2 \omega_\phi^n \leq C u_l(0) e^{-\theta t}.$$

Proof. We compute

$$\begin{aligned} \frac{\partial u_l(t)}{\partial t} &= 2 \frac{1}{V} \int_M (\nabla_l \dot{\phi}, \nabla_l \ddot{\phi}) + \frac{1}{V} \int_M |\nabla^l \dot{\phi}|^2 \Delta \dot{\phi} \omega_\phi^n \\ &\quad - \frac{1}{V} \int_M g_\phi^{i_1 j_1} \dots g_\phi^{i_{p-1} j_{p-1}} \dot{\phi}^{i_p j_p} g_\phi^{i_{p+1} j_{p+1}} \dots g_\phi^{i_l j_l} \nabla_{i_1 \dots i_l} \dot{\phi} \nabla_{j_1 \dots j_l} \dot{\phi} \omega_\phi^n. \end{aligned}$$

Since (6.5) gives $\ddot{\phi} = \Delta \dot{\phi} + \dot{\phi} + X(\dot{\phi})$, we use the Hölder inequality to estimate the first term

$$-2u_{l+1}(t) + \epsilon u_{l+1}(t) + C(\epsilon) u_l(t).$$

By (6.10), the second term is bounded by $u_l(t)$. Using (6.5) in the third term we get

$$-\frac{1}{V} \int_M g_\phi^{i_1 j_1} \dots g_\phi^{i_{p-1} j_{p-1}} (-Ric^{i_p j_p} + g_\phi^{i_p j_p} + X(\dot{\phi})^{i_p j_p}) g_\phi^{i_{p+1} j_{p+1}} \dots g_\phi^{i_l j_l} \nabla_{i_1 \dots i_l} \dot{\phi} \nabla_{j_1 \dots j_l} \dot{\phi} \omega_\phi^n$$

and by (6.9) we have the third term is controlled by $u_l(t)$. Combining theses three estimates, we then obtain by the L^p interpolation inequality

$$\begin{aligned} \frac{\partial u_l(t)}{\partial t} &\leq (-2 + \epsilon)u_{l+1}(t) + (C(\epsilon) + 2)u_l(t) \\ &\leq (-2 + \epsilon)u_{l+1}(t) + (C(\epsilon) + 2)(\delta u_{l+1}(t) + C(\delta)u_0(t)) \\ &\leq (C(\epsilon) + 2)C(\delta)u_0(t) \end{aligned}$$

provided $-2 + \epsilon + (C(\epsilon) + 2)\delta < 0$. We thus obtain by (6.20)

$$(6.22) \quad \frac{\partial u_l(t)}{\partial t} \leq Ce^{-\theta t}.$$

Since $u_l(\psi(t)) = \frac{1}{V} \int_M |\nabla^l \dot{\psi}|^2 \omega_\psi^n \rightarrow 0$ and $u_l(\psi(t))$ is invariant under the holomorphic transformation $\varsigma^* \varrho^{-1*}$, the lemma follows by integrating (6.22) from t to ∞ . \square

Since the Sobolev constant and the Poincare constant are uniformly bounded. Accordingly, (6.21) implies by the Sobolev imbedding theorem

$$(6.23) \quad |\dot{\phi}|_{C^1(g_\phi)} \leq Ce^{-\theta t}.$$

We apply $\phi(t) = \phi(0) + \int_0^t \dot{\phi} dt$ to obtain

$$(6.24) \quad |\phi(t)|_{C^0} \leq |\phi(0)|_{C^0} + Ce^{-\theta t}.$$

Consider the equation

$$\log \frac{\omega_\phi^n}{\omega^n} = \dot{\phi} - \phi - X(\phi) - a(t).$$

Yau's computation in [42] and its parabolic adaption in Cao [7] shows:

$$\begin{aligned} (\Delta_\phi - \frac{\partial}{\partial t})(e^{-C\phi}(n + \Delta\phi)) &\geq e^{-C\phi} \{C(\dot{\phi} - n)(n + \Delta\phi) \\ &+ (C + \inf_{i \neq k} R_{i\bar{i}k\bar{k}})(n + \Delta\phi) \frac{n}{n-1} e^{\frac{-(\dot{\phi} - \phi - X(\phi) - a(t))}{n-1}} + \Delta(-\phi - X(\phi)) - n^2 \inf_{i \neq k} R_{i\bar{i}k\bar{k}}\}. \end{aligned}$$

As shown in Tian-Zhu [38], at the maximal point of $e^{-C\phi}(n + \Delta\phi)$ where we get $\Delta\phi_i = C(n + \Delta\phi)\phi_i$, it follows $\Delta X(\phi) = X_{ki}\phi_{k\bar{i}} + CX(\phi)(n + \Delta\phi) \leq C(n + \Delta\phi)$. As a result, we apply (6.14), (6.12), (6.15) and (6.24) to obtain

$$0 < n + \Delta\phi \leq C,$$

which implies g and g_ϕ is L^∞ equivalent. Then by using Calabi's computation [4] (see Yau [42]) and the method for dealing with the extra term $\Delta X(\phi)$ in Zhu [43] we have C^3 norm of ϕ has uniform bound. Moreover, Theorem 2.8 implies all higher order derivatives are uniformly bounded and g and g_ϕ is C^∞ equivalent. So (6.23) gives

$$|\phi - \phi_\infty|_{C^1(g_\infty)} \leq C_1 e^{-\theta t}.$$

Finally, we have the exponential convergence of the modified Kähler-Ricci flow.

Proposition 6.10. *If the Kähler-Ricci flow converges to a Kähler-Ricci soliton in Cheeger-Gromov sense. Assume the initial Kähler potential is K_X -invariant, then the modified Kähler-Ricci flow must converge exponentially to a unique Kähler-Ricci soliton nearby.*

We remark here Zhu [44] also discussed the stability of Kähler-Ricci flow near a Kähler-Ricci soliton by using Perelman's estimate [30] and Chen-Tian's energy method [13][14].

7. WEAK FLOW

We can weaken the initial condition according to Chen-Tian [15], Chen-Tian-Zhang [16] and Song-Tian [36]. Let $a(t) = 0$ in (2.3), the potential equation reads

$$(7.1) \quad \begin{cases} \frac{\partial \varphi}{\partial t} &= \log \frac{\omega^n}{\omega^n} + \varphi \\ \varphi(0) &= \varphi_0. \end{cases}$$

We defined φ_0 is the limit of $\varphi_s \in PSH(M, \omega) \cap L^\infty(M)$ in L^∞ norm. Meanwhile, $\omega_{\varphi_0} \geq 0$ in the current sense. Let the weak solution be a limit of a sequence of approximate solution by

$$\varphi(t) = \lim_{s \rightarrow 0} \varphi(s, t).$$

In their articles, they proved that

Theorem 7.1. (*Chen-Tian [15], Chen-Tian-Zhang [16], Song-Tian [36]*) *If φ_0 is defined above with $|\varphi_0|_{L^\infty} \leq A$ and $|\frac{\omega^n}{\omega^n}|_{L^p(M, \omega)} \leq B$ for $p > 1$, there is a unique smooth solution $g_\varphi(t)$ of (1.1) for $t > 0$ such that*

$$\lim_{t \rightarrow 0^+} \varphi(t) = \varphi_0.$$

The key estimate in their proof is that (see Proposition 3.2 in Song-Tian [36])

$$(7.2) \quad |\varphi(t)|_{C^k} \leq C(t, T, k, A, B) \text{ on } (0, T].$$

Introduce the space

$$\mathcal{N}(\epsilon_0; B, p) = \{\varphi \mid |\varphi|_{L^\infty} \leq \epsilon_0, |\frac{\omega^n}{\omega^n}|_{L^p(M, \omega)} \leq B \text{ for some } p > 1\}$$

for fixing B and p . Here B and p should be chosen such that $\mathcal{N}(\epsilon_0; B, p)$ is not a empty set. Clearly, if $|\varphi_0|_{C^{1,1}} \leq \epsilon_0$, then $\varphi_0 \in \mathcal{N}(\epsilon_0, 1 + (2^n - 1)\epsilon_0, \infty)$. Actually, we can see that

Lemma 7.2. *When we fix $t_0 \in (0, T]$, for any $\epsilon_1 > 0$, there is a small ϵ_0 , for any $\varphi_0 \in \mathcal{N}(\epsilon_0; B, p)$, we have $|\varphi(t_0)|_{C^{2,\alpha}} \leq \epsilon_1$.*

Proof. If the conclusion fails, we could choose a sequence of φ_s such that

$$|\varphi_s|_{L^\infty} \leq \frac{1}{s} \text{ and } |\frac{\omega^n}{\omega^n}|_{L^p(M, \omega)} \leq B.$$

But for each corresponding solution $\varphi_s(t)$ constructed by Theorem 7.1, we have

$$(7.3) \quad |\varphi_s(t_0)|_{C^{2,\alpha}} > \epsilon_1.$$

Setting $g_{a\varphi_i\bar{j}} = \int_0^t (g_{i\bar{j}} + a\varphi_{i\bar{j}})da > 0$, we rewrite (7.1) as follows

$$\begin{cases} \frac{\partial \varphi}{\partial t} &= \Delta_{g_{a\varphi}} \varphi + \varphi \\ \varphi_s(0) &= \varphi_s. \end{cases}$$

By using the maximal principle we obtain that

$$(7.4) \quad \sup_M |\varphi_s(t_0)| \leq e^{t_0} \sup_M |\varphi_s|.$$

By (7.2), we can pass a subsequence of $\varphi_{s_i}(t_0)$ such that

$$\lim_{i \rightarrow \infty} \varphi_{s_i}(t_0) = \varphi_\infty(t_0)$$

in C^k for $k \geq 0$. Let $s = s_i$ in (7.4) then the limit approaches

$$(7.5) \quad \sup_M |\varphi_\infty(t_0)| \leq 0$$

which contradicts (7.3). \square

Now as we have a $C^{2,\alpha}$ small initial datum $\varphi(t_0)$, we normalize it to be $\varphi_0 - I(\varphi_0)$ which is also $C^{2,\alpha}$ small. Then we can solve equation (2.3) with this initial datum. Therefore combining Proposition 3.1, Proposition 4.5, Proposition 5.2 and Lemma 7.2 we obtain main Theorem 1.2. Analogously, we apply Proposition 6.5, Proposition 6.10 and Lemma 7.2 to obtain Theorem 1.1.

8. ANOTHER CHOICE OF HOLOMORPHIC TRANSFORMATIONS

In this section, we follow the argument by Bando-Mabuchi [3] and Chen-Tian [13] to find a good holomorphic transformation. I and J functional are defined as

$$I(\omega, \omega_\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega_\varphi^n),$$

$$J(\omega, \omega_\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \int_M \frac{i+1}{n+1} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}.$$

From Aubin [1] they are both semi-positive functionals and satisfy

$$(8.1) \quad 0 \leq I(\omega, \omega_\varphi) \leq (n+1)(I(\omega, \omega_\varphi) - J(\omega, \omega_\varphi)) \leq nI(\omega, \omega_\varphi).$$

for all $\varphi \in \mathcal{H}$. Fix $\varphi \in \mathcal{H}_0$, consider a functional

$$\Psi(\sigma) = (I - J)(\omega_\varphi, \sigma^* \omega) = (I - J)(\omega_\varphi, \omega_\rho)$$

for any $\sigma \in \text{Aut}_r(M)$ which is the reductive subgroup of $\text{Aut}(M)$ and $\sigma^* \omega = \omega + \partial \bar{\partial} \rho$. Since ω_ρ is a Kähler-Einstein metric, it satisfies

$$(8.2) \quad \log \frac{\omega_\rho^n}{\omega^n} + \rho = 0 \text{ and } I(\rho) = 0.$$

If ω_ρ is the minimal point of Ψ , for any $u \in \Lambda_1(\omega_\rho)$, we have

$$(8.3) \quad \int_M (\rho - \varphi) u \omega_\rho^n = 0.$$

It is known that $\eta(M) \cong \Lambda_1(\omega)$ for any Kähler-Einstein metric ω in [29]. In order to prove the minimizer of Ψ can always be attained, it is sufficient to prove

Proposition 8.1. *For all $\rho \in \{\rho | \sigma^* \omega = \omega_\rho, \sigma \in \text{Aut}_r(M), \Psi(\sigma) \leq r\}$, we have*

$$|\varphi - \rho|_{C^{2,\alpha}(g_\varphi)} \leq C(|\varphi|_{C^{4,\alpha}}).$$

Proof. Clearly,

$$-\Delta_\varphi(\rho - \varphi) < n \text{ and } -\Delta_\rho(\rho - \varphi) > -n.$$

Since the lower bound of the Green function is given by

$$(8.4) \quad G_\varphi \geq -\gamma \frac{D_\varphi^2}{\text{Vol}_\varphi} \doteq -A_\varphi.$$

Here the volume is a constant in fixed Kähler class and $\text{diam}(g_\varphi) \leq C \text{diam}(g)$ by $|\varphi|_{C^2} \leq C$. Using the Green formula and (8.4), we obtain

$$\begin{aligned} \sup_M(\rho - \varphi) &= \frac{1}{V} \int_M (\rho - \varphi) \omega_\varphi^n - \frac{1}{V} \int_M \Delta_\varphi(\rho - \varphi)(y)(G_\varphi(x, y) + A_\varphi) \omega_\varphi^n(y) \\ (8.5) \quad &\leq \frac{1}{V} \int_M (\rho - \varphi) \omega_\varphi^n + n A_\varphi. \end{aligned}$$

Similarly, we deduce

$$\begin{aligned} \inf_M(\rho - \varphi) &= \frac{1}{V} \int_M (\rho - \varphi) \omega_\rho^n - \frac{1}{V} \int_M \Delta_\rho(\rho - \varphi)(y)(G_\rho(x, y) + A_\rho) \omega_\rho^n(y) \\ (8.6) \quad &\geq \frac{1}{V} \int_M (\rho - \varphi) \omega_\rho^n - n A_\rho. \end{aligned}$$

Because $\text{Ric}(\rho) = \omega_\rho$, $\text{diam}(g_\rho) \leq \sqrt{2n-1}\pi$ by Myers theorem. Combining (8.5) and (8.6) we have

$$(8.7) \quad \text{Osc}_M(\rho - \varphi) \geq \frac{1}{V} \int_M (\rho - \varphi)(\omega_\varphi^n - \omega_\rho^n) + C(|\varphi|_{C^2}).$$

From (8.1) we obtain

$$\frac{1}{V} \int_M (\rho - \varphi)(\omega_\varphi^n - \omega_\rho^n) = I(\omega_\varphi, \omega_\rho) \leq (n+1)(I - J)(\omega_\varphi, \omega_\rho) \leq (n+1)r.$$

Since ω_ρ is a Kähler-Einstein metric, we have

$$(8.8) \quad (\omega_\varphi + \sqrt{-1}\partial\bar{\partial}(\rho - \varphi))^n = e^{-(\rho - \varphi) + h_\varphi} \omega_\varphi^n.$$

with

$$\sqrt{-1}\partial\bar{\partial}h_\varphi = \text{Ric}(\omega_\varphi) - \omega_\varphi \text{ and } \int_M e^{h_\varphi} \omega_\varphi^n = \text{Vol}(M).$$

By using the second order estimate in Yau [42], we get

$$\begin{aligned} n + \Delta_\varphi(\rho - \varphi) &\leq e^{C \text{Osc}_M(\rho - \varphi)} C(\sup_M(\inf_{i \neq k} |R_{\varphi i \bar{i} l \bar{l}}|), \inf_M S_\varphi, \sup_M h_\varphi) \\ (8.9) \quad &\leq e^{C \text{Osc}_M(\rho - \varphi)} C(|\varphi|_{C^4}). \end{aligned}$$

Then the Krylov estimate shows $\rho - \varphi$ has $C^{2,\alpha}$ bound. \square

Thus we also obtain the uniform bound of gauge ρ . Our previous discussion implies:

Corollary 8.2. *If $|\varphi|_{C^{4,\alpha}}$ is bounded and ρ is the minimizer of Ψ , then $|\varphi - \rho|_{C^{2,\alpha}}$ and $|\rho|_{C^{2,\alpha}}$ are both bounded.*

This lemma implies g_ρ is equivalent to g . We now turn to obtain the uniqueness of the critical points of the functional Ψ when φ is small. The second variation of Ψ at ρ is given by the formula

$$(8.10) \quad D^2\Psi_\rho(u, v) = \frac{1}{V} \int_M (1 + \frac{1}{2}\Delta_\rho \rho) uv \omega_\rho^n.$$

Lemma 8.3. *For all $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$ and $u \in \Lambda_1(\omega_\rho)$, the bilinear form $D^2\Psi_\rho(u, u)$ is positive. Hence ρ is unique.*

Proof. Note that (8.8) can be rewritten as

$$(8.11) \quad (\omega_\rho + \sqrt{-1}\partial\bar{\partial}(\varphi - \rho))^n = e^{-(\varphi - \rho) - h_\varphi} \omega_\rho^n.$$

By definition, h_φ is given by

$$h_\varphi = -\log \frac{\omega_\varphi^n}{\omega^n} - \varphi - \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n\right).$$

We conclude that

$$(8.12) \quad |h_\varphi|_{C^{2,\alpha}(g_\rho)} \leq C\epsilon_1 \leq \delta$$

by assumption of φ . Let

$$C_\perp^{2,\alpha}(M) = \{\varphi \in C^{2,\alpha}(M) \mid \int_M \varphi u \omega_\rho^n, \forall u \in \Lambda_1(\omega_\rho)\}.$$

Define the operator of (8.11) by

$$\begin{aligned} \Phi(a, b) &= \log \frac{(\omega_\rho + \sqrt{-1}\partial\bar{\partial}a)^n}{\omega_\rho^n} + a + b, \\ C_\perp^{2,\alpha}(M) \times C^\alpha(M) &\rightarrow C^\alpha(M). \end{aligned}$$

It is clear that $\Phi(\varphi - \rho, h_\varphi) = 0$ from (8.11). The linearized operator of (8.11) at $(a, b) = (0, 0)$ is given by

$$\delta_a \Phi(v) = \Delta_\rho v + v.$$

We infer that $\delta_a \Phi$ is invertible from $C_\perp^{2,\alpha}(M)$ to $C_\perp^\alpha(M)$. The implicit function theorem implies there is a small δ neighborhood of 0 in $C^\alpha(M)$ such that when $|h_\varphi|_{C^{2,\alpha}(g_\rho)} \leq \delta$, we have from (8.3) that

$$(8.13) \quad |\varphi - \rho|_{C^{2,\alpha}(g_\rho)} \leq C\delta.$$

Hence we deduce that

$$(8.14) \quad |\rho|_{C^{2,\alpha}} \leq |\varphi - \rho|_{C^{2,\alpha}} + |\varphi|_{C^{2,\alpha}} \leq C\epsilon_1 < 1$$

by using Corollary 8.2, (8.12) and choosing appropriate ϵ_1 . \square

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